Gust Response for Flat-Plate Airfoils and the Kutta Condition

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The case of a flat plate airfoil encountering a gust in incompressible flow is analyzed, with the generalization that the shed vorticity in the wake as well as the incident gust can convect at arbitrary velocity relative to the freestream. Bound vorticity and loading are calculated. At high frequency the bound vorticity is shown to be equivalent to an image of the incident gust, and drifts with the gust, except near the leading edge, and if the gust velocity is unequal to the wake velocity, near the trailing edge. When the wake moves relative to the freestream, there is a loading on the wake which forces nonzero loading at the trailing edge. A recent vortex chopping model is shown to depend explicitly on this fictitious wake loading, raising questions about the resulting noise prediction. A clarification in the application of the Kutta condition is given for the case of high frequency gust disturbances when both the gust and the wake drift with the freestream, showing that for inviscid flow the Kutta condition is satisfied automatically as the gust passes the trailing edge by allowing the flow to continue as if the edge were not present.

Nomenclature

b	= semichord
$E(\cdot)$	= combination of Fresnel integrals defined by Eq. (48)
F_w	= force on the wake
G_0	= amplitude of quasisteady circulation for sinusoidal time variation; introduced in Eq. (9)
g	= amplitude of wake vorticity for sinusoidal time variation case
$H_n^{(2)}(\cdot)$	= Hankel function
i	= $\sqrt{-1}$
$J_n(\cdot)$	= Bessel function of the first kind
$J_x(\cdot)$	= combination of Bessel functions; $J_0(\cdot)-iJ_1(\cdot)$
$K_n(\cdot)$	= modified Bessel function of the second kind
k,k_g,k_w	= $\omega b/U$, $\omega b/U_g$ or $\omega b/U_w$, respectively
$\mathbf{k}$	= wavevector
L	= lift
L_0, L_1	= lift produced by γ_0 or γ_1, respectively
$L_{q.s.}$	= quasisteady lift
L_s	= $2\pi b\rho_0 U w_0 e^{i\omega t}$; lift when $k=k_g=k_w \rightarrow 0$;
Δp	= pressure jump across the airfoil (lower minus upper)
$S(\cdot)$	= Sears function
t	= time
U	= freestream velocity
U_g, U_w	= convection velocity of the gust disturbance or the wake, respectively
u	= axial perturbation velocity
u_v, v_v	= chordwise and spanwise velocity components of an incident vortex
v	= spanwise velocity
w_g	= z velocity produced by gust
w_i	= z velocity induced by airfoil to cancel w_g
w_0	= amplitude of w_g
x,z	= Cartesian coordinates with x along chord and z normal to surface; normalized by b
Γ	= circulation about the airfoil
Γ_0, Γ_1	= integral of γ_0 or γ_1, respectively, over the airfoil
γ_0, γ_1	= quasisteady and wake induced vorticity on the airfoil, respectively
γ_g	= amplitude of incident vortex sheet
γ	= vorticity
ρ_0	= density of the fluid
ω	= circular frequency in airfoil fixed coordinates
Ω	= vorticity

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Introduction

AN analysis is presented extending the classical airfoil theory of von Kármán and Sears[1] and of Kemp[2] for incompressible inviscid flow to the case where the shed vorticity convects with arbitrary velocity, not necessarily the freestream velocity as is the usual case. Analysis along this line has been made by Howe,[3,4] but the present analysis gives a fuller account, especially with regard to a critical difficulty with the model which is the appearance of a force on the wake if required to move at other than the local velocity, which for the present case is the freestream velocity; this difficulty was not addressed by Howe. The force on the wake is evident from the fact that a vortex moving with respect to an infinite inviscid fluid experiences a force normal to the direction of motion. This can be made more rigorous by considering a thin control volume enclosing 1/2 wavelength of the shed vorticity (so that there is a net vorticity enclosed by the volume) and drifting at the speed of the vorticity. Since the wake is assumed frozen, the problem is steady, and the force can be calculated knowing the velocities and, from the momentum equation, the pressure. Since the control volume is thin, the ends can be ignored. The momentum terms give no contribution and the integral of the pressure over the upper and lower surfaces shows the wake forces to be proportional to the product of the enclosed circulation and the relative velocity.

The existence of this wake force suggests that the model is inaccurate. Nevertheless, it is felt that the analysis is enlightening by showing how subtleties in the model can create unintended effects in the result. Thus, even though in an actual flow the near wake may travel at other than the freestream velocity, care must be taken with the model to insure that there is no force on the wake. In order to correctly model a wake travelling at other than the freestream velocity, one needs a more complete wake description than simply moving the shed vorticity in a zero-thickness layer at the desired velocity.

For flat plate airfoil theory the application of the Kutta condition at the trailing edge requires the loading to be continuous with that in the wake. This is shown in the present article and is also stated by Basu and Hancock[5] and Sears.[6] Thus, when the wake loading is nonzero, the trailing edge loading cannot be zero. For a flat plate airfoil with zero loading on the wake, the paper shows that the specification of finite velocity at the trailing edge is equivalent to the specification that the loading on the trailing edge be zero. In particular, a discontinuity in either the loading or the bound vorticity implies a discontinuity in the other, and thus an infinite velocity, since a discontinuity in the strength of a vortex sheet produces an infinite velocity. This was discussed in some detail by Sears;[6] Basu and Hancock[5] also discuss this point, but mainly for the more complex problem of airfoils with finite trailing edge angles. This principle sheds light on the problem of vorticity in the freestream convected past the trailing edge. The papers of Ffowcs and Hall,[7] Jones,[8] Crighton and Leppington[9] and Howe[10] all treat the problem of noise from sources near a trailing edge. Howe[10] correctly predicts that no noise is generated by a vortex convecting with the freestream past the edge in the linear approximation, but the analysis is rather detailed, although it suited Howe's purpose and it is good to have a formal proof. The same conclusion can readily be reached by considering a steady vortex far upstream of the trailing edge in a coordinate system fixed with the vortex. Then the problem is essentially steady with zero ambient flow. Thus, any induced pressure must be second order in the vortex strength, and on reaching the trailing edge the condition of zero loading on the trailing edge is satisfied automatically; further discussion is given by Amiet.[11] This principle was likely well understood when airfoil gust response theory was being developed, since the problem of an airfoil entering a sharp edged gust[1] gives a distinct response when the gust strikes the leading edge, but no such response when it strikes the trailing edge. Also, to first order no surface pressure convects with the incident gust.

This principle was important in the author's formulation[12,13] of trailing edge noise. After realizing that the approach used to calculate leading edge noise, a velocity disturbance incident on an edge, did not work for the trailing edge problem, the author noticed a similarity to the leading edge problem if the pressure, rather than the velocity, was taken as the incident disturbance. Because of the general difficulty in calculating the surface pressure produced by convecting gusts, it was taken as the input in the trailing edge noise theory. Thus, while the leading edge problem is specified as an incident velocity disturbance with a zero velocity boundary condition on the airfoil, the trailing edge problem is specified as a incident pressure disturbance with a zero pressure boundary condition on the wake. This comment is made only to give the reasoning behind the author's formulation and should not overshadow the fact that a surface pressure formulation was used earlier by Chase[14] and Chandiramini,[15] which the author became aware of after completing the analysis.

The article also shows calculations of the bound vorticity on the airfoil, both for the classical Sears case and for the present extension of arbitrary wake velocity. The bound vorticity distribution is generally not studied in detail in airfoil theories; it may be utilized in the derivation, but it is seldom calculated explicitly. In retrospect, the results are not surprising but the principles discovered are important in helping to understand the gust response problem and the relevance of the Kutta condition.

The behavior of the vorticity is perhaps most easily understood in the high frequency limit. There is a leading edge effect, resulting in the creation of the bound vorticity. As the gust convects downstream, away from the leading edge, the expression for the bound vorticity settles down to what can be described as an image vorticity in an infinite plate that opposes and drifts with the incident gust. Martinez[16] notes that "at high frequencies the inviscid far wake shed by the airfoil continues to cancel the vertical velocity of the input gust." Howe[10] also notes that fact, but does not pursue it to the conclusions presented here. Interpreting the bound vorticity as an image vorticity is helpful in deciding the applicability of the Kutta condition as the disturbance passes the trailing edge. Thus, whereas one might think that the high frequency gust interacting with the trailing edge is a severe test of the Kutta condition, once it is clear that the flow upstream of the trailing edge reaches a steady state consisting of an incident gust and an image vorticity in the airfoil that drifts with the gust, but with no pressure pattern drifting with the gust, then it is also clear that, barring any viscous influence, the trailing edge has no effect on this steady state if the wake convects at the same speed as the incident gust. The gust and its image vorticity merely pass into the wake, perhaps implying that the Kutta condition forced the shedding of the vorticity, but in reality the vorticity was created upstream during the interaction of the gust with the leading edge region. For incompressible flow it appears that a gust striking the airfoil leading edge, rather than the trailing edge, is a greater test of the Kutta condition since it is during this period, in the process of setting up the image vortex pattern, that the most dynamic behavior of the flow around the airfoil occurs with the largest pressure gradient near the trailing edge. For compressible flow there is a time lag after the gust strikes the leading edge before vortex shedding can begin, since the disturbance must propagate from the leading to the trailing edge. As pointed out by one of the reviewers, classical airfoil theory is not strictly valid during the period when the gust strikes the leading edge because of viscous influences at the leading edge, but whatever theory one applies, the question of whether the Kutta condition is applicable or not is likely the most critical during this period.

The analysis also serves the useful purpose of determining to what extent the trailing-edge noise and vortex chopping noise predictions of Howe[3,4] depend on this wake force when the wake is required to move at other than the freestream velocity. In a private communication with the author, after the present analysis had been completed, Howe states that his theory is intended to represent the case of an airfoil with a finite thickness wake since the wake is specified to move at other than the freestream velocity even though he then approximates it as a zero-thickness layer in the plane of the airfoil. The control volume argument given above shows that a wake force exists at low frequency, even for a finite thickness wake. However, at high frequency, for gust wavelengths comparable to the wake thickness, the argument breaks down since the control volume can no longer be made long compared to its thickness, and the end effects that were ignored now become significant. At these high frequencies one can no longer conclude that shed vorticity moving at other than the freestream velocity in a finite thickness layer is subject to a force. However, one must note that any force induced by the shed vorticity on the airfoil is highly subject to the detailed movements and positions of the vorticity as it sheds into the wake, and it is unlikely that the specification by Howe of this shed vorticity in a zero thickness layer is representative of the actual situation. This is especially true of the vorticity in the near wake, within a wavelength of the edge, which is the most important in determining the force on the airfoil. The author does not wish to imply that there are no trailing-edge noise mechanisms, but rather that a more careful analysis is needed to properly represent them. The present analysis places the high frequency case in better perspective since it gives calculations, correct even at high frequency, of the induced airfoil loading and bound vorticity based on the shed vorticity assumed by Howe. The calculations show a trailing-edge force at all frequencies. However, the high frequency regime is most likely to contain some effect of the flow in the immediate vicinity of the trailing edge and the purpose of the present paper is to raise questions that may need a more detailed understanding of the motion of the vorticity in the vicinity of the edge to answer. For the frequency range below the typical von Kármán shedding frequency a strong argument can be given that the model does not behave properly to allow calculation of trailing-edge noise when the wake moves at other than the freestream velocity.

The simplest expression of this argument is that airfoil noise for a compact source is closely dependent on airfoil loading, and for arbitrary wake convection velocity the present model gives an analytically spurious distortion of the loading in just the region of importance for trailing-edge noise. The analysis in the paper gives calculations of the trailing-edge loading and shows that for gust wavelengths significantly smaller than a chord the airfoil lift consists of two terms, a term generated when a gust strikes the leading edge and a term generated when a gust srikes the trailing edge. The leading-edge term (Eq. (45)) is the normal Sears type of response, but the trailing-edge term, which is only present if the wake moves at other than the freestream velocity, depends directly on the change in loading between a point somewhat upstream of the trailing edge, in what was above called the steady state region of the bound vorticity, and a point at the trailing edge. The direct dependence of the noise on the fictitious trailing-edge loading gives strong evidence that the model gives a misleading prediction of the trailing-edge noise.

It should also be mentioned that there are theories that do not force the loading on the trailing edge or in the wake to zero when the wake is curved and has finite thickness; the curvature leads to a finite pressure jump across the wake. Sears,[19] Spence[20] and Brown and Stewartson[21] all give some discussion of this effect. However, the pressure jump is related to the boundary layer properties and the wake curvature, not just to the convection velocity of the wake. As noted by Sears,[19] "the problem of estimating boundary-layer properties for given, unsteady, potential flow is considerably more difficult than its steady flow counterpart." The detailed mathematical treatments by Spence and Brown and Stewartson are for the steady flow case.

The present analysis also gives some clarification of the relation of the acoustic analogy method used by Howe to more standard methods. It is shown herein that the equations given by Howe[4] for the far-field sound of an airfoil cutting a vortex are easily manipulated into the identical forms derived earlier by the author,[17,18] with the sole exception that the airfoil response function used by the author includes noncompactness and compressibility effects, whereas Howe assumes incompressible flow and allows the wake velocity, U_w, to differ from the freestream velocity, U. Although Howe's response function becomes identical with the von Kármán and Sears[1] result when $U_w = U$, in the form given by Howe the function does not produce the correct low frequency limit of steady airfoil theory when $U_w \neq U$ due to the inclusion of the force on the wake, and should not be considered a simple extension of the Sears function. A form is given herein that has the proper low frequency limit, but it may be of little value because of the finite trailing edge force.

Review of Unsteady Airfoil Theory for Incompressible Flow

The Analysis of von Kármán and Sears

A brief summary of the techniques and relevant equations for incompressible flow is presented first. The summary will closely follow the presentation of von Kármán and Sears,[1] who assumed that both the gust and the shed vorticity drift with the freestream. Generalizations are made where necessary to allow the possibility of the incident gust and shed vorticity moving at arbitrary velocity. The basic geometry is shown in Fig. 1. In the most general case treated, the airfoil, situated between $-1 \leq \times \leq 1$ in a freestream of velocity U, encounters a gust moving with velocity U_g; the airfoil shed vorticity in the wake convects downstream with velocity U_w. The length scales x and z are normalized by the semichord, b.

The vorticity on a flat plate airfoil produced by an upwash, either that from an airfoil motion or from a gust, can be

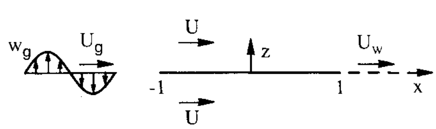

Fig. 1 Sinusoidal gust incident on a flat-plate airfoil.

divided into two parts. The first part, γ_0, is due to the quasi-steady airfoil upwash; in essence, it is the vorticity that would exist if the shed vorticity could be instantly transported to infinity. The second part, γ_1, is the vorticity induced on the airfoil by the shed vorticity. The expression given by von Kármán and Sears[1] for the vorticity, $d\gamma_1(x)$, on the airfoil at x produced by an element of vorticity, $\gamma(\xi)\, d\xi$, in the wake at ξ is

$$d\gamma_1(x) = \frac{1}{\pi}\frac{\gamma(\xi)d\xi}{\xi - x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} \tag{1}$$

The vorticity $\gamma_1(x)$ is then given by the integral of ξ over the wake, $1 < \xi < \infty$.

The total circulation $d\Gamma_1$ induced by the wake element $\gamma(\xi)d\xi$ is found by integration over the airfoil, giving

$$d\Gamma_1 = b\int_{-1}^{1} d\gamma_1(x)dx = b\gamma(\xi)d\xi\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) \tag{2}$$

The induced circulation Γ_1 due to the entire wake is

$$\Gamma_1 = b\int_1^\infty \left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right)\gamma(\xi)d\xi \tag{3}$$

The total circulation Γ on the airfoil is then

$$\Gamma = \Gamma_0 + \Gamma_1 = \Gamma_0 + b\int_1^\infty \gamma(\xi)\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right)d\xi \tag{4}$$

The pressure on the airfoil can be determined from the x component of the momentum equation

$$\rho_0\left(b\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0 \tag{5}$$

Since u on the upper surface is related to γ by $\gamma = \Delta u = u^- - u^+ = -2u^+$, the airfoil loading, Δp, (lower minus upper surface pressures) can be written

$$\frac{\Delta p(x,t)}{\rho_0} = -b\frac{\partial}{\partial t}\int_{-1}^{x}\gamma(\eta)d\eta - U\gamma(x) \tag{6}$$

The lift is then

$$\frac{L}{\rho_0} = b\int_{-1}^{1}\frac{\Delta p(x)}{\rho_0}dx = -b^2\frac{\partial}{\partial t}\int_{-1}^{1}\left[\int_{-1}^{x}\gamma(\eta)d\eta\right]dx - U\Gamma \tag{7}$$

Interchanging the order of the x and η integrations gives

$$\frac{L}{\rho_0} = -b^2\frac{d}{dt}\int_{-1}^{1}\gamma(\eta)d\eta\int_{\eta}^{1}dx - U\Gamma$$

$$= -b\frac{d\Gamma}{dt} + b^2\frac{d}{dt}\int_{-1}^{1}\eta\gamma(\eta)d\eta - U\Gamma \tag{8}$$

This last equation was given by Sears and Kuethe.[22] The derivation of the lift was not performed here in quite the manner of von Kármán and Sears[1] who use an overall momentum balance. In particular, a wake moving at other than the freestream velocity has a force on it which could accidentally be included in the calculation of the airfoil force if care is not taken. Von Kármán and Sears do not encounter this problem since they assume a wake traveling at the freestream velocity, while the above derivation does not make this assumption.

Sinusoidal Time Dependence

Because of its utility in Fourier analysis, one of the most fundamental cases to consider is that of a sinusoidal variation

$$\Gamma_0 = G_0 e^{i\omega t} \tag{9}$$

where G_0 is a constant. The vortex strength in the wake is similarly expressed as

$$\gamma(\xi) = g e^{i(\omega t - k_w \xi)} \tag{10}$$

where $k_w = \omega b / U_w$. Note that the wake moves at velocity U_w which is permitted to differ from the freestream velocity U. This is a generalization of the classic analysis presented by von Kármán and Sears[1] and others, but it has been considered by Howe.[3,4] Equation (4) gives for the total circulation

$$\frac{\Gamma(t)}{e^{i\omega t}} = G_0 + gb \int_1^{\infty} \left(\sqrt{\frac{\xi+1}{\xi-1}} - 1 \right) e^{-ik_w \xi} \mathrm{d}\xi \tag{11}$$

The unknown g is eliminated by taking the time derivative and noting that the rate of change of Γ must be equal and opposite to the rate at which vorticity is shed into the wake; i.e., $\mathrm{d}\Gamma/\mathrm{d}t = -\gamma(1)U_w$. This ensures conservation of vorticity at the trailing edge, giving for the relation between G_0 and g

$$-\frac{G_0}{bg} = \int_1^{\infty} \left(\frac{1+\xi}{\sqrt{\xi^2-1}} - 1 \right) e^{-ik_w \xi} \mathrm{d}\xi + \frac{1}{ik_w} e^{-ik_w} \tag{12}$$

This is simplified using the relations for the Bessel functions K_0 and K_1

$$K_0(iz) = \int_1^{\infty} \frac{e^{-iz\xi}}{\sqrt{\xi^2-1}} \mathrm{d}\xi \tag{13}$$

given on page 958 of Gradshteyn and Ryzhik[23] and

$$K_1(z) = -\frac{\mathrm{d}}{\mathrm{d}z} K_0(z) \tag{14}$$

The derivation is simplified by noting that

$$\lim_{\epsilon \to 0} \int_1^{\infty} \left(\frac{\xi}{\sqrt{\xi^2-1}} - 1 \right) e^{-iz\xi - \epsilon\xi} \mathrm{d}\xi = \int_1^{\infty} \left(\frac{\xi}{\sqrt{\xi^2-1}} - 1 \right) e^{-iz\xi} \mathrm{d}\xi \tag{15}$$

The exponential decay factor allows the left-hand-side to be written immediately as $K_1(iz) - (iz)^{-1} e^{-iz}$. Equation (12) then becomes

$$-\frac{G_0}{bg} = K_0(ik_w) + K_1(ik_w) = \frac{1}{ik_w S(k_w)} \tag{16}$$

This equation defines the Sears function $S(\cdot)$ in terms of the Bessel functions.

Expressions for the Surface Loading

Whereas the vorticity on the airfoil is divided into two parts, the airfoil loading can be divided into the three parts, quasisteady, the apparent mass and the wake. An apparent mass contribution is not needed for the vorticity since the vorticity is determined by Laplace's equation, not the momentum equation which includes time derivatives. The quasisteady lift is the lift that an airfoil having the given upwash, assumed constant, would experience. It is related to the bound vorticity γ_0 by Eq. (8), but with the time derivatives deleted; the time derivative terms are part of the apparent mass contribution. The complete γ_0 contribution to the lift will be calculated later. Here a known quasisteady pressure result is used to determine the quasisteady bound vorticity, thus eliminating the need to solve the steady flow problem.

To counter the incident gust field, w_g, the airfoil induces an opposing velocity w_i. Garrick[24] gives for the quasisteady contributions to the loading for an upwash, w_i, at ξ (w_i is defined here as positive in the same direction used to define positive loading)

$$\frac{\mathrm{d}\Delta p_{q.s.}}{\rho_0 U} = -\frac{2}{\pi} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+\xi}{1+\xi}} \frac{w_i(\xi,t)}{\xi - x} \mathrm{d}\xi \tag{17}$$

This is the negative of the expression given by Garrick since loading is here considered as positive upward, in contrast to Garrick who defines loading as negative for positive lift. Noting that $\mathrm{d}\Delta p_{q.s.} = -\rho_0 U \mathrm{d}\gamma_0$, the vorticity $\mathrm{d}\gamma_0$ is given by

$$\mathrm{d}\gamma_0 = \frac{2}{\pi} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+\xi}{1-\xi}} \frac{w_i(\xi,t)}{\xi - x} \mathrm{d}\xi \tag{18}$$

Integrating Eq. (18) over the chord (over x), the quasisteady circulation for an upwash, w_i, at ξ is

$$\mathrm{d}\Gamma_0 = 2b \sqrt{\frac{1+\xi}{1-\xi}} w_i(\xi,t) \mathrm{d}\xi \tag{19}$$

Consider a sinusoidal gust upwash

$$w_g(x,t) = w_0 e^{i(\omega t - k_g x)} \tag{20}$$

moving at velocity U_g, where $k_g = \omega b / U_g$. This is canceled on the airfoil by an induced upwash

$$w_i(x,t) = -w_0 e^{i(\omega t - k_g x)} \tag{21}$$

Introducing this into Eq. (19) and integrating over $-1 < \xi < 1$ gives for the quasisteady circulation

$$\Gamma_0 = -2\pi b w_0 J_x(k_g) e^{i\omega t} \tag{22}$$

where J_x is a combination of Bessel functions defined in the Nomenclature. Thus, the amplitude in Eq. (9) is

$$G_0 = -2\pi b w_0 J_x(k_g) \tag{23}$$

It should be noted that this argument to determine G_0 is not dependent on having the incident gust or the wake travel at the freestream velocity since the quasisteady circulation is dependent on neither the properties of the wake nor the freestream velocity. Equation (22) was given by Kemp[2] in his analysis of the case of arbitrary gust velocity. This result for G_0 can be combined with Eq. (16), giving for g

$$g = 2\pi i w_0 k_w S(k_w) J_x(k_g) \tag{24}$$

Introducing these expressions for G_0 and g into Eq. (11) gives for the circulation

$$\Gamma = -2\pi b w_0 S(k_w) J_x(k_g) e^{i(\omega t - k_w)} = \Gamma_0 S(k_w) e^{-ik_w} \tag{25}$$

Both g and Γ are for the case of a sinusoidal incident gust moving at velocity U_g and a wake drifting at velocity U_w. The quasisteady lift is found by noting that $L_{q.s.} = -\rho_0 U \Gamma_0$. Thus,

$$L_{q.s.} = L_s J_x(k_g) e^{i\omega t} \tag{26}$$

where L_s is the lift in the long wavelength limit $k_g \to 0$ representing an airfoil at a constant angle of attack.

For the sinusoidal gust problem the quasisteady vorticity, γ_0, is given by Eq. (18) with w_i given by Eq. (21); the wake induced vorticity, γ_1, is given by Eq. (1) with $\gamma(\xi)$ given by Eq. (10) and g by Eq. (24). The vorticity field is completely determined by these equations. The distribution of loading over the chord can thus be determined using Eq. (6). In the most general case this leads to integrals that could not be evaluated in closed form, but when $U_g = U_w = U$ it is possible to evaluate these integrals. The algebra is somewhat involved; the result of substituting γ_0 into Eq. (6) gives for Δp_0

$$\frac{\Delta p_0}{\rho_0 U w_0 e^{i\omega t}} = 2\sqrt{\frac{1-x}{1+x}}\, J_0(k) + 2ik(\pi - \cos^{-1}x) J_x(k) \tag{27}$$

This should not be confused with the quasisteady loading, for which the lift is given by Eq. (26). By contrast, Eq. (27) contains apparent mass terms because of the inclusion of the time derivative in Eq. (6). In the same manner, substituting γ_1 into Eq. (6) gives for Δp_1

$$\frac{\Delta p_1}{\rho_0 U g e^{i\omega t}} = -\frac{\pi - \cos^{-1}x}{\pi S(k)} - \frac{1}{\pi}\sqrt{\frac{1-x}{1+1}} K_0(ik) \tag{28}$$

These can be added giving the total loading distribution and lift for a sinusoidal gust with $U_w = U_g = U$ as

$$\frac{\Delta p}{\rho_0 U w_0} = 2\sqrt{\frac{1-x}{1+x}} S(k) e^{i\omega t} \tag{29}$$

$$L/L_s = S(k) \tag{30}$$

Equation (30) was given by von Kármán and Sears.[1] Equation (29) can also be derived from results given, for example, by Garrick,[24] but the author believes that the present derivation is interesting because of its relative simplicity.

For the general case the vorticity γ is the sum $\gamma = \gamma_0 + \gamma_1$. For the case $U_w = U_g = U$ being considered, a simpler analytical form for γ can be found using the velocity potential, obtained from Eq. (6) after setting $u = \partial\phi/\partial x$; thus,

$$\phi = -\frac{b}{\rho_0 U}\int_{-\infty}^{x} p\left(\xi, y, t - \frac{b(x-\xi)}{U}\right) d\xi \tag{31}$$

Taking the x derivative and recalling that the vorticity is related to the axial velocity at the surface by $\gamma = -2u^+$ allows the vorticity to be written as

$$\rho_0 U \gamma(x,t) = -\Delta p + \frac{b}{U}\frac{\partial}{\partial t}\int_{-1}^{x} \Delta p[\xi, t - b(x-\xi)/U] d\xi \tag{32}$$

This equation can be considered the inverse of Eq. (6), giving the vorticity in terms of the airfoil loading, while Eq. (6) gives the airfoil loading in terms of the vorticity; both equations are general in that they are not explicitly dependent on the convection speed of either the incident gust or the wake. Substituting Δp from Eq. (6) into Eq. (32) leads to an identity after an interchange of order in the resulting double integral. Equations (29) and (32) give for the vorticity produced by a sinusoidal gust with both the gust and the shed vorticity drifting at the freestream velocity U

$$\frac{\gamma(x,t)}{w_0 e^{i\omega t} 2S(k)} = -\sqrt{\frac{1-x}{1+x}} + ike^{-ikx}\int_{-1}^{x}\sqrt{\frac{1-\xi}{1+\xi}} e^{ik\xi} d\xi \tag{33}$$

Although this cannot be evaluated in closed form, the result at the trailing edge, $x = 1$, can be found either by integration of Eq. (33) or from Eq. (25) by setting $-\gamma(1)U = d\Gamma/dt$.

Incident Gust and Shed Vorticity Moving at Arbitrary Velocity

The analysis of airfoil problems with an incident gust or wake moving at other than the freestream velocity is not new. Thus, Kemp[2] considers the case of the incident disturbance moving at arbitrary speed, but never the case of a wake moving at arbitrary speed. In Kemp's analysis, the incident upwash was assumed to be produced by adjacent airfoils that could sustain a force. Such a force originates from the difference in velocity immediately above and below a vortex that is moving along a horizontal line relative to the fluid. This also occurs for shed vorticity forced to move at other than the freestream velocity in an inviscid flow.

Howe[3,4] considers the case of arbitrary wake velocity. The model of vorticity moving rectilinearly at other than the freestream velocity in a zero thickness layer corresponds to a physical problem of a shear layer with a force imposed on it, producing a pressure jump across the shear layer. In this case the airfoil lift cannot be calculated from the change in momentum of the vortex distribution throughout the fluid, as done by von Kármán and Sears[1] for the case with no wake force, since this would include the wake forces.

Having a force applied to the wake also leads to a force at the trailing edge if one applies a Kutta condition for smooth flow there. The same point was also recognized by Basu and Hancock[5] who note that "...the loading across the trailing edge is zero for consistency with the condition of zero loading across the shed vorticity." This presents a problem in the calculation of trailing-edge noise. Airfoil noise can be described in terms of the fluctuation of force on the airfoil; in the model with shed vorticity moving at other than the freestream velocity the fact that the force does not go to zero at the trailing edge means that there will be a sudden change in force as an incident gust passes the trailing edge. The noise so produced is thought to be an artifact of the model, not an actual noise generation mechanism.

Amiet[25] also has considered the case of a sinusoidal gust moving at an arbitrary velocity, but with the wake convecting with the freestream. Equation (26b) of the airfoil analysis of Amiet[25] shows that the trailing-edge noise becomes zero for a sinusoidal upwash traveling at the freestream velocity. [Note that there is a minor, but critical, printing error in this equation. The factor $(k(\lambda - 1)$ should read $(k/\lambda - 1))$.] This can be explained by noting that an incident gust travelling with the freestream produces no pressure on the airfoil after moving downstream from the leading edge; thus, the Kutta condition is satisfied automatically when the gust passes the trailing edge, just as for the case of a vortex passing the trailing edge, discussed in the Introduction. Since a general upwash can be synthesized from sinusoidal gusts, the trailing-edge noise produced by a general gust, when both the gust and its shed vorticity convect with the freestream, is zero by this analysis. When one considers sources produced by and embedded in the boundary layer, the problem becomes more involved. Because of the difficulty in calculating the surface pressure due to convecting turbulence (see, e.g., Chase[26]) the trailing edge noise analysis of the author[12,13] assumes the convecting pressure disturbance is specified, rather than the gust upwash.

In order to illustrate these ideas, the pressure at the trailing edge will be calculated for the case of the shed vorticity moving at speed U_w. Equation (6) gives the airfoil loading distribution in terms of the airfoil vorticity. At the trailing edge ($x = 1$) the equation for the airfoil loading becomes

$$\frac{\Delta p(1,t)}{\rho_0} = -\frac{d\Gamma}{d\Gamma} - U\gamma(1) \tag{34}$$

Since the shed vorticity convects away from the trailing edge at velocity U_w, and since the vorticity strength must be continuous at the trailing edge for the Kutta condition of finite velocity to be satisfied, the rate at which vorticity is shed into the wake is $U_w\gamma(1)$. To conserve global vorticity, the rate of change of vorticity bound on the airfoil must be $d\Gamma/dt = -U_w\gamma(1)$. Substitution into Eq. (34) gives for the loading at the trailing edge

$$\frac{\Delta p(1,t)}{\rho_0} = (U_w - U)\gamma(1) \tag{35}$$

This shows that the force on the airfoil at the trailing edge is continuous with that across the wake, i.e., the wake vorticity moves at velocity $V_{\text{rel.}} = U_w - U$ relative to the ambient fluid, and the force on a moving unit segment of a vortex sheet is $F = \rho_0 V_{\text{rel.}} \gamma$, which at the trailing edge gives the same result as Eq. (35).

It is of interest to also calculate the overall lift force on the airfoil. This can be done by evaluating the three terms in Eq. (8). First the quasisteady component γ_0 of the vorticity is considered. This gives rise to a lift, denoted by L_0; this is not the same as the quasisteady component of lift $L_{q.s.}$ in Eq. (26) since in calculating L_0 from γ_0 the dynamical components are included. The first and third terms in Eq. (8) are obtained from Γ_0 in Eq. (22). The second term is evaluated by first calculating the x moment of $\mathrm{d}\gamma_0$ and then taking the ξ integral. The result is

$$\int_{-1}^{1} x\gamma_0(x)\mathrm{d}x = w_0 \frac{2\pi}{k_g} J_1(k_g) e^{i\omega t} \tag{36}$$

From Eq. (8) the overall lift L_0 induced by the quasisteady component of vorticity γ_0 is then

$$\frac{L_0}{2\pi b w_0 \rho_0 U e^{i\omega t}} = (1+ik)J_0(k_g) + \left[k + i\left(\frac{k}{k_g} - 1\right)\right] J_1(k_g) \tag{37}$$

This result is independent of the convection velocity of the wake.

The wake induced lift is found by substituting γ_1 (the integral of Eq. (1) over ξ) into Eq. (8). Performing the integration in Eq. (3), with the shed vorticity given by Eq. (10) and g by Eq. (24), gives

$$\Gamma_1 = -\frac{bg}{ik_w}\left[\frac{1}{S(k_w)} - e^{-ik_w}\right] \tag{38}$$

The first moment of the vorticity is

$$\int_{-1}^{1} x\gamma_1(x)\mathrm{d}x = \frac{g}{ik_w}\left[K_1(ik_w) - e^{-ik_w}\left(1 + \frac{1}{ik_w}\right)\right] \tag{39}$$

Combining Eq. (38) and (39) as prescribed in Eq. (8) gives for the wake induced lift

$$\frac{L_1}{b\rho_0 U e^{i\omega t}} = \frac{g}{ik_w}\left[ikK_1(ik_w) - \frac{1+ik}{S(k_w)} + \left(1 - \frac{k}{k_w}\right)e^{-ik_w}\right] \tag{40}$$

The total lift is then

$$\frac{L_0 + L_1}{L_s} = S(k_w)\left[ikK_1(ik_w) + \left(1 - \frac{k}{k_w}\right)e^{-ik_w}\right]J_x(k_g) + i\frac{k}{k_g}J_1(k_g) \tag{41}$$

This reduces to Eq. (30) when $U_w = U_g = U$. Also, when $U_w = U$ Eq. (41) reduces to Eq. (5) of Kemp.[2] The Bessel functions of imaginary argument may be recast in terms of the Hankel functions by the relations

$$K_0(ik) = -i\frac{\pi}{2}H_0^{(2)}(k), \qquad K_1(ik) = -\frac{\pi}{2}H_1^{(2)}(k) \tag{42}$$

The high frequency limit, $\omega \to \infty$, of these expressions is of interest since this allows leading and trailing-edge effects to be separated. A large-scale, low frequency disturbance interacts simultaneously with the leading and trailing edges, while a small-scale, high frequency disturbance interacts first with the leading edge, then the trailing edge. The asymptotic limits of the Bessel functions are

$$\lim_{k\to\infty} H_1^{(2)}(k) \sim i \lim_{k\to\infty} H_0^{(2)}(k) = i\sqrt{\frac{2}{\pi k}}e^{-i(k-\pi/4)}$$

$$\lim_{k\to\infty} S(k) = \frac{1}{\sqrt{2\pi k}}e^{i(k-\pi/4)}$$

$$\lim_{k\to\infty}\begin{bmatrix} J_0(k) \\ J_1(k)\end{bmatrix} = \sqrt{\frac{2}{\pi k}}\begin{bmatrix}\cos(k-\pi/4) \\ \sin(k-\pi/4)\end{bmatrix} \tag{43}$$

These give for the high frequency limits of the lift expressions L_0 and L_1

$$\frac{L_0}{L_s} \sim \frac{-L_1}{L_s} \sim ik\sqrt{\frac{2}{\pi k_g}}e^{-i(k_g - \pi/4)} \tag{44}$$

Thus, the lowest order terms of L_0 and L_1 cancel. The lowest order term of the sum $L_0 + L_1$ is

$$\frac{L_0 + L_1}{2\pi b w_0 \rho_0 U e^{i\omega t}} \sim \frac{e^{-i\pi/4}}{\sqrt{2\pi k_g}}\frac{U_g}{U}\left[e^{ik_g} + i\left(\frac{U_w}{U_g} - 1\right)e^{-ik_g}\right] \tag{45}$$

The first term on the right-hand-side is the high frequency limit of the Sears function, and corresponds to the leading edge, due to the factor $\exp(ik_g)$. The second term, with the factor $\exp(-ik_g)$ corresponds to the trailing edge. To see this, consider a localized incident gust, such as a very thin jet, with an upwash composed of Fourier components $w_0(k_g) = $ const. Then when taking the inverse Fourier transform of Eq. (45), the first term contributes a factor $\exp(i\omega t + ik_g) = \exp[ik_g(U_g t/b + 1)]$. This gives a response centered around the time, $t = -b/U_g$, that the jet strikes the leading edge. The second term gives a response centered about this time, $t = b/U_g$, that the jet passes the trailing edge, and this contribution is zero if $U_w = U_g$, since then the gust induced loading is constant on passing the trailing edge.

Thus, at high frequency the noise produced by the airfoil (which is proportional to $\mathrm{d}L/\mathrm{d}t$ for the compact case) contains the two terms in Eq. (45). The second term contains the factor $(U_w - U_g)$ which is the same factor found by Howe[3,4] for the case with the Kutta condition. In Ref. 4 for the case of an airfoil cutting a vortex, Howe assumes $U_g = U$, but allows $U_w \neq U$; this leads to a sound pulse as the vortex passes the trailing edge. The present analysis finds this model to be of questionable validity, but the case of high frequency, around the typical von Kármán shedding frequency of trailing-edge thickness/freestream velocity, needs further study to understand the actual flow in the trailing-edge region. The compressible flow analysis of Amiet[17,18] assumes $U_w = U$, and thus finds no sound as the vortex passes the trailing edge, but the author does not wish to imply that a more detailed analysis of the trailing edge behavior would fail to find any trailing-edge noise mechanism.

Consider the physical reason for the factor $U_w - U_g$ in the trailing edge component of Eq. (45). As noted above, the loading on a unit length of a vortex sheet γ moving at velocity U_g in a freestream U is $F = \rho_0(U_g - U)\gamma$. A given segment of the incident gust induces in the plate an image vortex segment γ moving at the same velocity U_g, and the force on the image vortex is $F_{\text{airfoil}} = \rho_0(U_g - U)\gamma$. As the vortex approaches and passes the trailing edge, its velocity changes to U_w and the force on it is now $F_{\text{wake}} = \rho_0(U_w - U)\gamma$. The change in force is $\Delta F = F_{\text{wake}} - F_{airfoil} = \rho_0(U_w - U_g)$; since noise generation is dependent on the time derivative of the force, the noise calculation will contain this factor. In the actual case the wake often convects at a velocity other than that of the freestream, but this is for finite thickness wakes that include the induced effects of adjacent vortices.

It is instructive to compare Eq. (41) above with the results of Howe[4] for the noise produced by a airfoil cutting a vortex. Although Howe's result might at first seem to be somewhat different from the sinusoidal gust lift response given in Eq. (41), for long wavelength or compact flow the sound is just the time derivative of the lift; also, Howe decomposes the vortex into its sinusoidal gust components, so that a direct comparison becomes possible. For this comparison, the total wake force will be added to Eq. (41) to give the total force on the fluid. The wake force $F_w = -\rho_0(U-U_w)\Gamma_w$ where Γ_w is the circulation about the entire wake, which must be the negative of the circulation about the airfoil Γ given by Eq. (25). If F_w is added to the total airfoil force in Eq. (41), F_w is found to cancel the term containing the factor $(1-k/k_w)$. After a slight rearrangement the result is

$$\frac{L_0+L_1}{L_s}+\frac{F_w}{L_s}=-i\frac{k}{k_w}\frac{H_1^{(2)}(k_w)}{H_0^{(2)}(k_w)-iH_1^{(2)}(k_w)}J_x(k_g)+i\frac{k}{k_g}J_1(k_g) \quad (46)$$

Setting $k_g=k$ in Eq. (46) reduces the expression to the case considered by Howe. If one takes the sum of Howe's[4] Eq. (3.26) and (3.27), taking the complex conjugate since Howe uses $\exp(-i\omega t)$ where the present author uses $\exp(i\omega t)$, one finds that this sum contains exactly the factor on the right hand side of Eq. (46). In addition to showing that Howe's result can be derived by classical airfoil principles, this also shows that Howe's expression for the noise includes the contribution from the wake force; this is not surprising since Howe did not perform a detailed integration of the surface loading in the manner presented here. This can be further verified by taking the limit $\omega\to 0$; noting that $H_1^{(2)}$ dominates over $H_0^{(2)}$ for small argument, one finds that $(L_0+L_1+F_w)/L_s\to U_w/U$ as $\omega\to 0$ in Eq. (46). In other words, one finds that on approaching the quasisteady case the result of Howe is still affected by the wake velocity; the wake force F_w does not go to zero as $\omega\to 0$. In contrast, taking the limit of Eq. (41) as $\omega\to 0$, one finds that $(L_0+L_1)/L_s\to 1$ as $\omega\to 0$, which is the expected result; the airfoil lift approaches the quasisteady lift and is not affected by the wake velocity as $\omega\to 0$. In the high frequency limit, one finds from the above results that the ratio $F_w/(L_0+L_1)\sim\omega^{-1/2}$ as $\omega\to\infty$ so that F_w gives a negligible contribution to the overall force on the fluid in the high frequency limit; in this limit it makes no difference whether the wake force is added to the calculated airfoil force, but the details of the vorticity distribution near the trailing edge nevertheless strongly affect the trailing edge loading. Thus, as noted in the Introduction, Howe's result does not appear to be valid below the general frequency given by wake thickness/wake velocity. Further understanding of the flow in the trailing-edge region is needed before definite conclusions can be reached for the high frequency range.

It is interesting to consider the order of accuracy of the loading calculation. Accepting the assumptions that the incident disturbance, the incident vortex, drifts with the freestream and that the shed vorticity lies on the axis, the calculation of the bound and shed vorticity by standard incompressible flow methods such as that of von Kármán and Sears[1] is exact. Thus, the fluid velocity is also known exactly, and the pressure could be found exactly, if desired, by indirect integration of the momentum equation. One's initial thoughts might be that there would be terms such as u^2 that a linearized pressure calculation, Eq. (6), neglects, and that this could lead to large errors in the loading results near the leading edge where $u\to\infty$. However, u^2 terms cancel when the difference between the upper and lower surface pressures is taken. From the momentum equation, neglecting no nonlinear terms, for an observer in a region of no vorticity, or exterior to the incident vortex core, in place of Eq. (6) one finds

$$\frac{\Delta p}{\rho_0}=-b\frac{\partial}{\partial t}\int_{-1}^{x}\Delta u\,\mathrm{d}x-(U+u_v)\Delta u-v_v\Delta v \quad (47)$$

where u_v and v_v are the chordwise and spanwise components of the incident disturbance and Δu and Δv are the jumps in chordwise and spanwise velocity across the airfoil. The only difference from Eq. (6) is the appearance of the two terms containing u_v and v_v. This can be expressed in a more intuitive form. For a two-dimensional problem the integral of Δu from -1 to x is just the circulation, $\Gamma(x)$, about this portion of the airfoil. The time derivative of this integral must be the rate at which vorticity is entering or leaving this region. Defining a convection velocity, U_b, of the bound vorticity as $U_b\gamma(x)=-\partial\Gamma(x)/\partial t$, Eq. (47) becomes simply $\Delta p/\rho_0=-(U+u_v-U_b)\gamma(x)$. Basically this equation is just an expression of the relation *airfoil loading* $=-\rho_0 V_{\rm rel}\gamma$ where the velocity of the vorticity relative to the local fluid velocity $V_{\rm rel}$ now includes the local velocity u_v of the incident vortex. u_v will generally be small compared to U, except perhaps near the vortex core, but it must be included by a nonlinear theory. In the present case, neglecting u_v and v_v in Eq. (47) is the same order of approximation as assuming that the incident vortex drifts with the freestream in the presence of the airfoil.

As an aside, without going into great detail, further comparisons can be made between the vortex chopping analysis of Howe[4] and the earlier analysis of Amiet,[17,18] in order to determine any difference between the two solutions. Equations (3.26) and (3.27) of Howe give the far-field sound in terms of an integration of the Fourier transformed *vorticity* disturbance multiplied by the sinusoidal gust incompressible flow lift expression given by Eq. (46) above; Eq. (24) of Amiet[17] gives the far-field sound in terms of an integration of the Fourier transformed *velocity* disturbance multiplied by the lift expression for compressible flow. By replacing the compressible flow lift expression in Amiet's analysis with Eq. (46) above for incompressible flow, all that remains in order to make a comparison between these equations is to use the relation between the Fourier components of the incident vorticity field, $\Omega(\boldsymbol{k})$ and the incident velocity field, $\boldsymbol{v}(\boldsymbol{k})$ where $\boldsymbol{k}$ is the wavevector of the transformed field. It is relatively straightforward to show from the definition of vorticity (see for example Amiet,[27] Eq. (B10)) that this relation is

$$\boldsymbol{v}(\boldsymbol{k})=i\frac{\boldsymbol{k}\times\Omega(\boldsymbol{k})}{k^2} \quad (48)$$

With this relation, the result of Howe is seen to be identical to that of Amiet. One can also compare the explicit expression for the corresponding Fourier components of vorticity and velocity for the given upwash field of an incident vortex. Since Howe uses the same vortex description as Amiet, one would expect the Fourier components to be equivalent. Again using Eq. (48), Eq. (4.3) of Howe[4] for the Fourier components of the incident vorticity field is found to lead directly to Eq. (38) of Amiet[17] for the Fourier component of the incident gust velocity field. The same conclusion is drawn by comparison of the corresponding forms for the jet velocity, given by Eq. (4.8) of Howe[4] and Eq. (6) of the earlier paper of Amiet.[18] Stated differently, by taking the zero Mach number limit of the results of Amiet,[17,18] one obtains the results of Howe, except for the case $U_w\neq U$ which is analyzed here. Thus, Howe's[4] designation of the solution of Amiet[17] as a numerical solution may be somewhat misleading. Whereas Amiet makes a numerical "exact" evaluation of the final intractable integral, Howe uses a high frequency approximation to derive an analytical expression. Amiet also gives a simple high frequency approximation for the case of an airfoil encountering a delta function gust, which was shown to be algebraically equivalent to the vortex problem.

Bound Vorticity at High Frequency

There appears to be some uncertainty in the literature[3,9,10] whether or not to apply the Kutta condition for short wavelength disturbances convecting at the freestream velocity past the trailing edge. While this question may need a more complete viscous analysis to understand the case of an airfoil with a finite thickness trailing edge at high frequency, the present discussion shows that for a flat plate airfoil with a sharp trailing edge the Kutta condition is satisfied almost by default when the vortex convects with the freestream. In essence the flow with embedded vorticity far upstream of the edge adjusts to the presence of the plate, inducing image vorticity in the plate, as shown below. This flow pattern produces no pressure on the airfoil for linearized flow. Thus when the embedded vorticity reaches the trailing edge, the imposition of the Kutta condition makes no change in the flow; the wake is a natural continuation of the image vorticity, and no noise is radiated. This argument depends on the gust and its image remaining in equilibrium and requires the neglect of nonlinear effects such as the image gust affecting the motion of the incident gust; it also is modified to some extent by viscous forces, but this is not expected to create a significant deviation from the behavior of the inviscid linear solution. One should be careful to distinguish between the above case of vorticity convected with the freestream and a case in which an oscillatory flow is imposed on the airfoil trailing edge, such as from an incident acoustic wave or from a gust convecting at other than the free stream velocity. The case of a gust convecting at other than the freestream velocity produces a loading on the airfoil that must disappear as one approaches the trailing edge; it is this forcing of the loading to zero at the trailing edge that tests the validity of the Kutta condition.

In fact, one can argue that the case where no Kutta condition is applied (meaning that no vortex shedding is allowed) is not a physically realistic condition for the case of convected disturbances past a sharp trailing edge. Once the upstream flow has "relaxed" to the presence of the convected disturbance, to say that no vorticity can be shed requires an induced force on the airfoil, producing additional vorticity to cancel that which is naturally convecting past the trailing edge.

The above statements are shown by calculating the bound vorticity from Eq. (32) in the high frequency limit for the Sears case ($U_g = U_w = U$), using the pressure given by Eq. (29). With the change of variables $\xi = -\cos\theta$ this becomes

$$\frac{\gamma(x)}{2w_0 S(k) e^{i\omega t}} = -\sqrt{\frac{1-x}{1+x}} + ike^{-ikx}\int_0^{\cos^{-1}(-x)} (1+\cos\theta)e^{-ik\cos\theta}\mathrm{d}\theta \tag{49}$$

The major contribution to the integral for large k is from the region $\theta \sim 0$. The cosine in the exponential can be expanded as $\cos\theta \sim 1-\theta^2/2$ giving, for large k,

$$\frac{1}{2S(k)e^{i\omega t}}\lim_{k\to\infty}\left[\frac{\gamma}{w_0}\right] = -\sqrt{\frac{1-x}{1+x}} + 2\sqrt{\pi k}\, e^{-ik(x+1)}E\left(\frac{k}{2}[\cos^{-1}(-x)]^2\right) \tag{50}$$

where $E(\cdot)$ is the combination of Fresnel integrals

$$E(a) = \frac{1}{\sqrt{2\pi}}\int_0^a e^{it}\frac{\mathrm{d}t}{\sqrt{t}} \tag{51}$$

Since $S(k)\to 0$ for $k\to\infty$ as in Eq. (43), the first term in Eq. (47) goes to zero for x not near the leading edge. For x not too near the leading edge, the Fresnel integral becomes $(1+i)/2$ and

$$\lim_{k\to\infty}\gamma = 2iw_0 e^{i(\omega t - kx)} \qquad \text{for } x \neq -1 \tag{52}$$

It will be noted from Eq. (47) that there is a leading edge effect, but no trailing edge effect on the bound vorticity; that is, away from the leading edge the bound vorticity approaches the constant amplitude given by Eq. (52) and this is maintained even at the trailing edge. The imposition of a Kutta condition produces no disturbance at the trailing edge, resulting in the flow pattern that one might expect to occur naturally. That is, if the vorticity can convect off the trailing edge, creating no flow perturbation, while satisfying all the boundary conditions of no flow through the surface, one would expect it to do so.

The incident upwash in Eq. (20) can be produced by a sinusoidal sheet of vorticity convecting with the flow. For a sheet of vorticity defined as

$$\gamma(x) = \gamma_g e^{i(\omega t - k_g x)} \qquad -\infty < x < \infty \tag{53}$$

the w, or normal, velocity distribution produced at the sheet can be shown to equal that of Eq. (20) if $\gamma_g = -2iw_0$, showing that $\gamma_g = -\gamma$ from Eq. (52). Thus, for high frequency the bound vorticity in the airfoil at positions not near the leading edge is just the mirror image of the incident vorticity producing the upwash. This was also noted by Martinez.[16] The flow has adjusted to the presence of the airfoil by producing a mirror image vorticity in the airfoil. This generally means an unsteady loading and the shedding of vorticity as the incident flow disturbance passes the leading edge, but once the adjustment is made, no further adjustment is needed as the flow disturbance passes the trailing edge; both the incident vorticity and the mirror image vorticity pass unimpeded into the wake.

It is interesting to determine whether it is the quasisteady vorticity, γ_0, or the wake vorticity, γ_1, that produces the asymptotic limit given in Eq. (52). One might expect this limit to be produced entirely by γ_0, which is almost correct, but the slight deviation from this expected result is worth noting since it can lead to some confusion. Equation (18) gives an expression for γ_0 by taking an integral over ξ with the principal part of the integral assumed. The induced upwash is assumed to be that of Eq. (21). With this and the assumption that x is not near the leading or trailing edges, it can be shown that the primary contribution to the ξ integral comes from the region $\xi = x$. The integral then reduces to

$$\lim_{k_g\to\infty}\left[\frac{\gamma_0(x)}{w_0 e^{i\omega t}}\right] = \frac{2}{\pi}\oint_{-1}^{1} e^{-ik_g\xi}\frac{\mathrm{d}\xi}{x-\xi} = \frac{4i}{\pi}e^{-ik_g x}\int_0^\infty \sin(k_g\xi)\frac{\mathrm{d}\xi}{\xi} = 2ie^{-ik_g x} \quad \text{for } x \neq \pm 1 \tag{54}$$

This is independent of U_w, and the gust convection velocity U_g is not necessarily equal to U. Equation (54) agrees with the expression for γ in Eq. (52) for $U_g = U$, except near $x=1$ which is excluded from the range of Eq. (54), but not from the range of Eq. (52). In fact, $\gamma_0 \to 0$ as $x\to 1$, from simple steady airfoil theory. An expression for γ_1 is found by integrating Eq. (1) over ξ from 1 to ∞ with $\gamma(\xi)$ given by Eq. (10). Substituting $x = 1-\epsilon$ and $\xi = 1+\epsilon\eta$ gives

$$\frac{\gamma_1(1)}{ge^{i\omega t}} = \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\frac{e^{-ik_w}}{\sqrt{2+\epsilon}}\int_0^\infty\sqrt{\frac{2+\epsilon\eta}{\eta}}\,e^{-ik_w\epsilon\eta}\frac{\mathrm{d}\eta}{\eta+1}\right] = \frac{2}{\pi}e^{-ik_w}\int_0^\infty\frac{\mathrm{d}\zeta}{\zeta^2+1} = e^{-ik_w} \tag{55}$$

This applies or all k_w, and verifies that as one approaches the trailing edge from upstream, the vorticity is continuous with that in the wake. For $\omega\to\infty$, Eq. (24) gives $g\to 2iw_0(k_w/k_g)^{1/2}\exp[i(k_w-k_g)]$. Then, at $x=1$, $\gamma_1 = 2iw_0(k_w/k_g)^{1/2}\exp[i(\omega t - k_g)]$, which, for $U_g = U_1$, is the same result given by Eq. (52). In other words, on approaching the trailing edge γ_0 changes from $2iw_0\exp[i(\omega t-k_g)]$ to 0 while γ_1 changes from 0 to $2iw_0(k_w/k_g)^{1/2}\exp[i(\omega t-k_g)]$, the sum remaining constant when $U_g = U_w$, as given by Eq. (52).

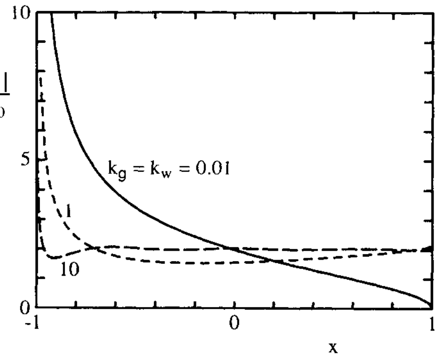

Fig. 2 Amplitude of the airfoil bound vorticity for three values of the wavenumber when wake and gust convection velocities are equal.

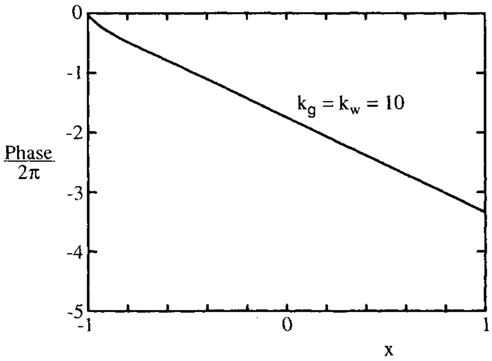

Fig. 3 Phase of the bound vorticity for the highest frequency case in Fig. 2.

In descriptive terms, the interaction of an airfoil with a small high frequency eddy leads to the production of mirror image vorticity in the airfoil. If the eddy has impinged on the leading edge, the formation of the image vorticity can be considered to be produced by the influence of the leading edge. This image vorticity is given by the quasisteady part γ_0 of the vorticity until the trailing edge is approached, whereupon $\gamma_0 \rightarrow 0$ and for $U_g = U_w$, γ_1 increases at an equivalent rate to keep constant the overall vorticity γ on the airfoil as seen by an observer drifting with the gust. It is important to recognize this trade-off between γ_0 and γ_1 as otherwise it is easy to misinterpret the increase in γ_1 near the trailing edge as a generation of vorticity by the trailing edge. This change-over is an artifact of the artificial splitting up of the vorticity into quasisteady and wake contributions.

Calculated Results

Explicit results for the bound vorticity and loading, calculated from the previous equations, are presented in Figs. 2-5. These figures illustrate many of the points made above. Figure 2 shows the amplitude of the bound vorticity for three values of the wavenumbers k_g and k_w, with $k_g = k_w$. Since the pressure is not given in this figure, the value of the wavenumber k is not needed. For $k_g = k_w = 0$ (a value of 0.01 was used for convenience in programming) the vorticity conforms to the typical quasisteady behavior given by $\gamma \sim [(1-x)/(1+x)]^{1/2}$. As k_g increases, the leading edge effect region becomes progressively smaller, with the vorticity more and more quickly reaching the asymptotic value for a semi-infinite plate. As noted above, $|g| \rightarrow 2w_0(k_g/k_w)^{1/2}$ for $\omega \rightarrow \infty$, so $|\gamma|/w_0 \rightarrow 2$ for $k_g = k_w$. The curve for $k_g = 10$ is very near this asymptotic value for values of x not near the leading edge.

Figure 3 gives the phase corresponding to Fig. 2 for the case $k_g = k_w = 10$. Except for a small region near the leading edge where bound vorticity is being created, the phase behaves linearly with x. This, combined with the fact that the amplitude in Fig. 2 is essentially constant, shows the propagating character of the bound vorticity. The wavenumber can be deduced from the slope in Fig. 3, and it agrees with that of the incident gust, showing that the bound vorticity is indeed an image vortex pattern travelling with the incident disturbance.

Figure 4 shows the results when $k_g \neq k_w$; for comparison, a repeat from Fig. 2 of the curve for $k_g = k_w = 1$ is given. When $k_g \neq k_w$, one notes a trailing-edge effect that is absent when $k_g = k_w$. This effect is emphasized even more in Fig. 5 which shows the loading for a higher frequency case. This loading was calculated by direct integration of the vorticity given by Eq. (6). For calculation of the loading the value of k is taken as $k = k_g$. The dashed plot with $k_g = k_w = k$ is the typical Sears case, and the loading is the usual $[(1-x)/(1+x)]^{1/2}$, going to zero at the trailing edge. When k_w is allowed to deviate from the value of k, one notes a significant trailing-edge effect, with the loading no longer becoming zero at the trailing edge. For this case Eq. (24) gives $|g| = 2.8$. The loading, which must be continuous with that in the wake for the Kutta condition to apply, is $|\Delta p| = \rho_0(U - U_w)|g|$; since U_w is taken as $U/2$, the loading at the trailing edge is $|\Delta p|/\rho_0 U w_0 = |g|/2 = 1.4$, agreeing with the value in Fig. 5.

Velocity and Loading Kutta Conditions

The Kutta condition at an airfoil trailing edge is specified in various manners in the literature. Some discussion is given by Basu and Hancock[5] regarding whether one should specify zero loading or finite velocity at the trailing edge. They note that a finite pressure difference at the trailing edge is unacceptable on physical grounds. They also note that "there is no analytic solution for an aerofoil undergoing an unsteady motion which satisfies both the condition of finite velocities and the condition of zero loading at the trailing edge." An unstated assumption is that attention is being directed to airfoils with finite trailing-edge angles, since they also state that such a solution is available for an airfoil with a cusped trailing edge. Here we consider the simplest case, that of a flat plate airfoil, which could be considered an airfoil with a cusped trailing edge. A discussion of the flat plate case is given by Sears.[6]

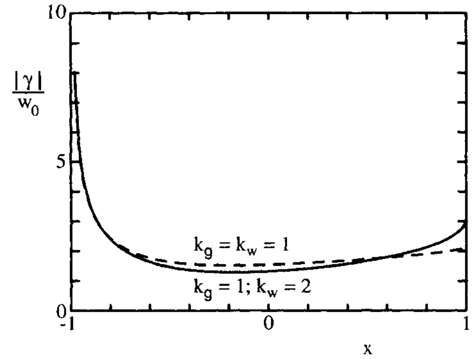

Fig. 4 Amplitude of the airfoil bound vorticity when the gust convection velocity is twice that of the wake, compared to the corresponding plot in Fig. 2.

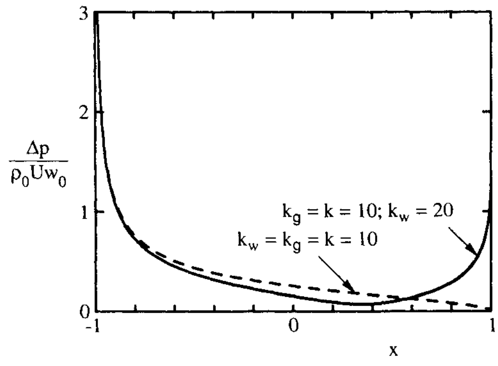

Fig. 5 Airfoil loading distribution when the gust convection velocity is twice that of the wake, compared to the corresponding plot when the two velocities are equal.

Kirchhoff's solution shows that the linearized solution for the case of an unsteady motion of a body in an inviscid fluid can be generated by a distribution of monopoles and dipoles on the surface. For a flat plate airfoil the monopoles are discarded since they combine with the monopoles on the opposite side to give a dipole. Thus, the solution can be generated by a distribution of surface dipoles, a dipole being equivalent to a force. With a little consideration of such dipole distributions and in particular, an analysis of the velocity field around a continuous dipole distribution that has a sudden spatial change in amplitude, such as would occur at a trailing edge with finite loading but with no loading supported by the wake, one realizes that an infinite velocity always occurs at such points. This is an inevitable consequence of the singular behavior of the dipole solution. Conversely, a dipole solution that approaches zero sufficiently rapidly at the edge, such as force $\sim x^{1/2}$ with x denoting the distance from the edge, with no force on the wake, gives a finite velocity at the edge. From this one can conclude that satisfying the condition of no force on the trailing edge is equivalent to satisfying the Kutta condition.

This can be put in more mathematical form using Eq. (6) and (32). To simplify these results slightly, introduce the dimensionless variables $\boldsymbol{t} = \mathrm{Ut/b}$, $\boldsymbol{x} = \mathrm{x}$, $\gamma = \gamma/\mathrm{U}$, $\Delta \boldsymbol{p}(\boldsymbol{t}) = \Delta p(t)/(\rho_0 U^2)$. Then Eqs. (6) and (32) can be combined to give

$$\frac{\partial}{\partial t}\int_{-\infty}^{x} \Delta p(\eta, t - x + \eta) - \gamma(\eta, t)]\mathrm{d}\eta = 0 \tag{56}$$

With a little study for small changes in x about a supposed discontinuity, it becomes clear that one cannot have a discontinuity in either Δp or γ without both having a discontinuity.

Conclusions

A detailed analysis has been made of the incompressible flow problem of a flat plate airfoil encountering a gust with both the gust and shed vorticity moving with arbitrary velocity. Whereas the case of gust moving at arbitrary velocity incident on an airfoil models a physically realistic problem, an inviscid treatment of an airfoil with shed vorticity moving in a zero thickness layer at arbitrary velocity implies a force on the wake, and thus raises questions about the model. This is especially important in the calculation of trailing edge noise since a physical argument shows that this noise is directly dependent on the change of force in passing the trailing edge. Calculations of the bound vorticity for a high frequency gust incident on an airfoil show that the vorticity shed into the wake is just the image vorticity generated upstream to satisfy the boundary condition of no flow through the airfoil. For this case the Kutta condition is satisfied almost by default if there is no change of force as the gust passes the trailing edge and to impose a condition of no vortex shedding is unrealistic. Finally, it appears that when the wake drifts with the freestream, the analysis of Howe for an airfoil cutting a vortex gives the same results as the zero Mach number limit of the earlier analysis of Amiet.

References

[1]von Kármán, T., and Sears, W. R., "Airfoil Theory of Non-Uniform Motion," *Journal of the Aeronautical Sciences,* Vol. 5, Aug. 1938, pp. 379–390.

[2]Kemp, N. H., "On the Lift and Circulation of Airfoils in Some Unsteady-Flow Problems," *Journal of the Aeronautical Sciences,* Vol. 19, Oct. 1952, pp. 713–714.

[3]Howe, M. S., "A Review of the Theory of Trailing Edge Noise," *Journal of Sound and Vibration,* Vol. 61, Dec. 1978, pp. 437–465.

[4]Howe, M. S., "Contributions to the Theory of Sound Production by Vortex-Airfoil Interaction, with Application to Vortices with Finite Axial Core Velocity Defect," *Proceedings of the Royal Society of London,* Vol. A420, Nov. 1988, pp. 157–182.

[5]Basu, B. C., and Hancock G. J., "The Unsteady Motion of a Two-Dimensional Aerofoil in Incompressible Inviscid Flow," *Journal of Fluid Mechanics,* Vol. 87, July 1978, pp. 159–178.

[6]Sears, W. R., "Unsteady Motion of Airfoils with Boundary-Layer Separation," *AIAA Journal*, Vol. 14, Feb. 1976, pp. 216–219.

[7]Ffowcs Williams, J. E., and Hall, L. H., "Aerodynamic Sound Generation by Turbulent Flow in the Vicinity of a Scattering Half Plane," *Journal of Fluid Mechanics,* Vol. 40, March 1970, pp. 657–670.

[8]Jones, D. S., "Aerodynamic Sound Due to a Source Near a Half-Plane," *Journal of the Institute of Mathematics and its Applications,* Vol. 9, Feb. 1972, pp. 114–122.

[9]Crighton, D. G., and Leppington, F. G., "Scattering of Aerodynamic Noise by a Semi-Infinite Compliant Plane," *Journal of Fluid Mechanics,* Vol. 43, Oct. 1970, pp. 721–736.

[10]Howe, M. S., "The Influence of Vortex Shedding on the Generation of Sound by Convected Turbulence," *Journal of Fluid Mechanics,* Vol. 76, Aug. 1976, pp. 711–740.

[11]Amiet, R. K., "Comment on the Surface Pressure Produced by an Airfoil-Gust Interaction," *Journal of Sound and Vibration*, Vol. 121, March 1988, pp. 571–572.

[12]Amiet, R. K., "Noise Due to Turbulent Flow Past a Trailing Edge," *Journal of Sound and Vibration,* Vol. 47, Aug. 1976, pp. 387–393.

[13]Amiet, R. K., "Effect of the Incident Surface Pressure Field on Noise due to Turbulent Flow Past a Trailing Edge," *Journal of Sound and Vibration,* Vol. 57, March 1978, pp. 305–306.

[14]Chase, D. M., "Sound Radiated by Turbulent Flow off a Rigid Half-Plane as Obtained from a Wavevector Spectrum of Hydrodynamic Pressure," *Journal of the Acoustical Society of America,* Vol. 52, Sept. 1972, pp. 1011–1023.

[15]Chandiramini, K. L., "Diffraction of Evanescent Waves; Applications to Aerodynamically Scattered Sound and Radiation from Unbaffled Plates," *Journal of Sound and Vibration,* Vol. 55, 1974, pp. 19–29.

[16]Martinez, R., "Predictions of Unsteady Wing and Pylon Forces Caused by Propeller Installation, NASA Contractor Rep. 178298, Langley Research Center, May 1987.

[17]Amiet, R. K., "Airfoil Gust Response and the Sound Produced by Airfoil-Vortex Interaction," *Journal of Sound and Vibration,* Vol. 107, June 1986, pp. 487–506.

[18]Amiet, R. K., "Intersection of a Jet by an Infinite Span Airfoil," *Journal of Sound and Vibration,* Vol, 111, Dec. 1986, pp. 499–503.

[19]Sears, W. R., "Some Recent Developments in Airfoil Theory," *Journal of the Aeronautical Sciences,* Vol. 23, May 1956, pp. 490–499.

[20]Spence, D. A., "Wake Curvature and the Kutta Condition," *Journal of Fluid Mechanics,* Vol. 44, Part 4, Dec. 1970, pp. 625–636.

[21]Brown, S. N. and Stewartson, K., "Wake Curvature and the Kutta Condition in Laminar Flow," *Aeronautical Quarterly,* Vol. 26, Nov. 1975, pp. 275–280.

[22]Sears, W. R., and Kuethe, A. M., "The Growth of the Circulation of an Airfoil Flying through a Gust," *Journal of the Aeronautical Sciences,* Vol. 6, July 1939, pp. 376–378.

[23]Gradshteyn, I. S., and Ryzhik, I. M., *Tables of Integrals, Series, and Products,* Academic Press, New York 1965.

[24]Garrick, I. E., "Nonsteady Wing Characteristics," Section F of *Aerodynamic Components of Aircraft at High Speeds,* Princeton Series in High Speed Aerodynamics and Jet Propulsion, Vol. VII, A. F. Donovan and H. R. Lawrence, eds., Princeton Univ. Press, Princeton, NJ, 1957, pp. 658–793.

[25]Amiet, R. K., "High Frequency Thin-Airfoil Theory for Subsonic Flow," *AIAA Journal,* Vol. 14, Aug. 1976, pp. 1076–1082.

[26]Chase, D. M., "The Character of the Turbulent Wall Pressure Spectrum at Subconvective Wavenumbers and a Suggested Comprehensive Model," *Journal of Sound and Vibration,* Vol. 112, Jan. 1987, pp. 125–146.

[27]Amiet, R. K., "Leading and Trailing Edge Noise from a Helicopter Rotor," United Technologies Research Center, Rep. UTRC 86-53, 1986.